

# THE CURVATURE HOMOGENEITY BOUND FOR LORENTZIAN FOUR-MANIFOLDS

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**ABSTRACT.** We prove that a four-dimensional Lorentzian manifold that is curvature homogeneous of order 3, or  $\text{CH}_3$  for short, is necessarily locally homogeneous. We also exhibit and classify four-dimensional Lorentzian,  $\text{CH}_2$  manifolds that are not homogeneous. The resulting metrics belong to the class of null electromagnetic radiation, type N solutions on an anti-de Sitter background. These findings prove that the four-dimensional Lorentzian Singer number  $k_{1,3} = 3$ , falsifying some recent conjectures[1]. We also prove that invariant classification for these proper  $\text{CH}_2$  solutions requires  $\nabla^{(7)}R$ , and that these are the unique metrics requiring the seventh order.

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## 1. INTRODUCTION

The invariant classification (IC) of spacetimes is central importance in general relativity. Let  $(M, g)$  be a pseudo-Riemannian manifold, and let  $R^i = \nabla^i R$  denote the  $i$ th-order covariant derivative of the Riemann curvature tensor. Throughout, we assume that the algebraic type of the curvature tensor and its covariant derivatives is constant. This means that the curvature and its covariant derivatives can be normalized to some standard form so that there is a well defined automorphism group  $G_i$  of  $R^i$ . We set  $N_i = \dim G_i$ . The general equivalence problem for pseudo-Riemannian geometry was solved by Elie Cartan, who proved that  $R, R^1, \dots, R^q$ , up to sufficiently high order classifies the metric up to a diffeomorphism [2]. Let  $q_M$  denote the smallest order required for the invariant classification (IC) of  $M$ . Cartan established the bound  $q_M \leq n(n+1)/2$ ; here  $n(n+1)/2$  is the dimension of the corresponding orthogonal frame bundle.

Motivated by applications to GR, Brans[3] and Karlhede [4] showed how to recast the equivalence problem in terms of differential invariants on the base manifold. One advantage of Karlhede's algorithm (see below) is that it improves the general IC bound to  $q_M \leq N_0 + n + 1$ . The IC algorithm was refined and implemented in a computer algebra system by MacCallum, Åman [5], and others [6]. See [7] for a recent review. Here is the algorithm, largely as it appears in [8, Section 9.2]. Let  $\eta_{ab}$  be a constant, non-degenerate quadratic form having the same signature as the metric  $g$ . Henceforth, we use  $\eta_{ab}$  to raise and lower frame indices. Let  $\mathcal{O}(\eta)$  denote the  $n(n-1)/2$  dimensional Lie group of  $\eta$ -orthogonal transformations, and say that a coframe  $\theta^a$  is  $\eta$ -orthogonal if

$$g = \eta_{ab} \theta^a \theta^b \quad (1)$$

### The Karlhede IC algorithm

1. Set  $q = 0, G_{-1} = \mathcal{O}(\eta), t_{-1} = 0$ . All  $\eta$ -orthogonal frames are permitted.

2. Compute  $R^q$  relative to a permitted  $\eta$ -orthogonal frame.
3. Determine  $G_q \subset G_{q-1}$ , the automorphism group of  $R^{(q)} := \{R^0, R^1, \dots, R^q\}$ .
4. Restrict the frame freedom to  $G_q$  by putting  $R^q$  into standard form (normalizing some components to a constant, for instance.)
5. Having restricted the frame freedom, the functions in the set  $R^{(q)}$  are differential invariants. Let  $t_q$  be the number of independent functions over  $M$  in  $R^{(q)}$ .
6. If  $N_q < N_{q-1}$  or  $t_q > t_{q-1}$ , then increase  $q$  by one, and go to step 2.
7. Otherwise, the algorithm terminates. The differential invariants in  $R^{(q-1)}$  furnish essential coordinates. The isometry group has dimension  $n - t_{q-1} + N_{q-1}$ . The orbits have dimension  $n - t_{q-1}$ . The integer  $q_M = q$  is the IC order.

In principle, the invariant classification and the equivalence problems are solved at step 7 because the essential coordinates obtained via the algorithm allow the metric to be expressed in a canonical form that incorporates the other differential invariants as essential constants and essential functional parameters.

An optimal bound on  $q_M$  where  $M$  is a Lorentzian, 4-dimensional manifold is of particular interest in classical general relativity. The well-known Petrov-Penrose classification of the Weyl tensor gives  $N_0 = 0$  for Petrov types I, II, III;  $N_0 \leq 2$  for types N and D; and  $N_0 \leq 3$  for type O<sup>1</sup>. Hence,  $q_M \leq 5$  for types I, II, III;  $q_M \leq 7$  for Petrov types N, D; and  $q_M \leq 8$  for type O. These bounds have been improved, and it is now known that  $q_M \leq 6$  for a type D spacetime [9], and  $q_M \leq 6$  for a type O spacetime [10].

The question of whether the 7th order bound for type N spacetimes was sharp or whether it could be improved remained open for over 20 years. Recently, the present authors exhibited a family of type N exact solutions for null electromagnetic radiation on an anti-de Sitter background for which  $q_M = 7$ , and thereby established that Karlhede's bound of  $q_M \leq 7$  was indeed sharp [11]. In the present paper, we give a detailed derivation of the exact solutions in question, and prove that these metrics are, essentially, the unique spacetimes for which  $q_M = 7$ .

Our approach is to consider the restricted IC problem for the class of proper, curvature homogeneous geometries and to express the curvature homogeneity condition in terms of an appropriate set of field equations.

**1.1. Curvature homogeneity and invariant classification.** A pseudo-Riemannian manifold is *curvature homogeneous* of order  $k$ , or  $\text{CH}_k$  for short, if the components of the curvature tensor and its first  $k$  covariant derivatives are constant relative to some choice of frame. We say that  $M$  is *properly*  $\text{CH}_k$  if it belongs to class  $\text{CH}_k$ , but does not belong to class  $\text{CH}_{k+1}$  [12]. The  $\text{CH}$  class includes all homogeneous geometries, because a homogeneous space is curvature homogeneous to all orders. Thus, a (locally) homogeneous manifold is  $\text{CH}_k$  for all  $k$ , but not properly  $\text{CH}_k$  for any  $k$ . The following remarkable result was originally proved by Singer in the Riemannian context [13] and extended to arbitrary signatures in [14].

**Theorem 1.1** (Singer, Podesta and Spiro). *For every signature  $(a, b)$ , there exists an integer  $k$ , such that if  $M$  is  $\text{CH}_k$ , then necessarily  $M$  is locally homogeneous.*

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<sup>1</sup>Here, one has to consider the possible symmetries of the Ricci tensor.

Following Gilkey[1], we use  $k_{a,b}$  to denote the smallest such integer  $k$ . The proof of the theorem utilizes an integer invariant  $k_M$ , defined to be the smallest  $k$  such that  $N_k = N_{k+1}$  [15, 12]. Singer established the following.

**Theorem 1.2** (Singer's criterion). *If  $M$  is curvature homogeneous of order  $k_M + 1$ , then  $M$  is locally homogeneous.*

Consequently, if  $M$  is properly  $\text{CH}_k$ , then necessarily,  $k \leq k_M$ . It follows that  $k_{a,b} = \max\{k_M + 1\}$  where  $M$  runs over the class of proper curvature-homogeneous manifolds of signature  $(a, b)$ . Also note that Singer's criterion follows as a special case of the Karlhede algorithm. Indeed,  $M$  is a homogeneous space if and only if all differential invariants are essential constants (the structure constants of the corresponding Lie algebra.) Thus,  $M$  is a homogeneous space if and only if  $t_{k_M+1} = 0$ . The latter condition is equivalent to  $M$  being curvature homogeneous of order  $k_M + 1$ .

As we will show, the class of proper CH manifolds plays a key role in the search for geometries with a maximal  $q_M$ . Already in [9], Collins and d'Inverno showed that the conditions for an IC order of  $q_M = 7$  are very stringent. Without naming it as such, their necessary conditions (shown below) suggest a proper  $\text{CH}_2$  geometry.

- (C1) The components of the curvature tensor must be constants.
- (C2) The invariance group at zeroth order  $G_0$ , must have dimension 2.
- (C3) The dimension of the invariance group and the number of functionally independent components must not both change on differentiating.
- (C4) We must produce at most one new functionally independent component on differentiating.
- (C5) The dimension of the invariance group must go down by at most one dimension on differentiating.

These conditions imply that  $q_M = 7$  can be achieved if

$$(t_0, t_1, t_2, t_3, t_4, t_5, t_6, t_7) = (0, 0, 0, 1, 2, 3, 4, 4); \quad (2)$$

$$(N_0, N_1, N_2, N_3, \dots) = (2, 1, 0, 0, \dots). \quad (3)$$

It is conceivable that a  $q_M = 7$  might be achieved with a different sequence of  $t_i$  and  $N_i$ , but that would require  $N_i = N_{i+1} > N_{i+2}$  for some  $i$ . Such a phenomenon is called pseudo-stabilization and it is known to be highly atypical[16]. Since  $N_{k_M} = N_{k_M+1}$ , we can also say that the curvature automorphism groups pseudo-stabilize if there exists an  $i > k_M + 1$  such that  $N_i < N_{k_M}$ . If we exclude the possibility of pseudo-stabilization then the Collins–D'Inverno conditions describe a proper, curvature homogeneous geometry.

**Proposition 1.3.** *Suppose that there exists an  $M$  such that  $q_M = N_0 + n + 1$ , i.e. the Karlhede bound is sharp. Also, suppose that the curvature automorphism groups do not pseudo-stabilize. Then,  $M$  is properly  $\text{CH}_k$  where  $k = N_0$ .*

In the case of type N spacetimes, if the Karlhede bound  $q_M \leq 7$  really were sharp, and if we exclude the possibility of pseudo-stabilization, then we are forced to consider the existence of a proper  $\text{CH}_2$  geometry.

To put it another way, the value of Gilkey's integer  $k_{1,3}$  is crucial, because if  $k_{1,3} \leq 2$ , then a proper  $\text{CH}_2$  Lorentzian manifold does not exist. Let us review what is known about bounds on  $k_M$ . In his original paper [13] Singer established the bound  $k_M < n(n-1)/2$ ; here the right-hand side is the dimension of the

orthogonal group. In the Riemannian case, Gromov asserts that  $k_M < \frac{3}{2}n - 1$  [17]. More generally, Gilkey and Nikčević[18] have shown that  $k_{a,b} \geq \min(a, b)$ . It is known that there are no proper  $\text{CH}_1$  Riemannian manifolds in 4 dimensions [19]. In the 3-dimensional, Lorentzian case, proper CH geometries have been classified [20, 21] and it is known that  $k_{1,2} = 2$  [15]. In the 4-dimensional, Lorentzian case proper  $\text{CH}_1$  manifolds were shown to exist in [22]. The recent book by Gilkey [1] has additional references, and examples of higher-dimensional proper CH manifolds of general signature.

Regarding the quantity  $k_{1,3}$ , Gilkey has conjectured that  $k_{1,3} = 2$ , and more generally that  $k_{a,b} = \min(a, b) + 1$  [1]. However, in the present paper, we establish that these conjectures are false by showing that  $k_{1,3} = 3$ . We do this by proving that in the 4-dimensional, Lorentzian case a proper  $\text{CH}_3$  metric does not exist, and by classifying and exhibiting all proper  $\text{CH}_2$  metrics. Equations (135)-(138) give the proper  $\text{CH}_2$  exact solution as a null-orthogonal tetrad. All proper  $\text{CH}_2$  spacetimes belong to this family, which depends on two essential constants and one function of one variable. We also show that this family is a specialization of type N exact solutions for coupled gravity and electromagnetic waves propagating in anti-de Sitter background, first described in [23] and [24]. Further analysis reveals that, generically, there are no Killing vectors, but that there is a singular subcase with an  $\text{SL}_2\mathbb{R}$  isometry group and another singular subfamily with a 1-dimensional isometry group. Finally, we prove that the generic, proper  $\text{CH}_2$  metrics satisfy the Collins–D’Inverno conditions and enjoy the remarkable property of  $q_M = 7$ .

**1.2. The CH field equations.** A methodology for expressing and analyzing the field equations for a CH geometry is essential to our investigation. Previously, Estabrook and Wahlquist described vacuum solutions[25] as an involutive exterior differential system( EDS) on the bundle of second-order frames. Our approach is to formulate the necessary field equations as an EDS using two-forms and commutator quantities (equivalently, connection components) as canonical variables. Unlike the field equations for vacuum, the field equations for CH spacetimes are, in general, overdetermined, with integrability condition that manifest as algebraic constraints on the curvature and connection scalars. Our result is proved by deriving integrable configurations for the CH field equations corresponding to various algebraic types of the curvature tensor, and by using Singer’s criterion to rule out the homogeneous subcases. We will use this method to classify proper  $\text{CH}_1$  four-dimensional, Lorentz geometries in a forthcoming publication.

Section 2 of the present paper introduces the necessary field variables required to formulate the CH field equations. Section 3 recasts these CH equations as an EDS, and introduces the crucial concept of a CH-configuration, the algebraic data that underlies a CH geometry. The actual classification and the proof of our main result is found in Section 4. The final section contains some concluding remarks.

## 2. THE CH EQUATIONS

**2.1. Preliminaries.** A homogeneous space is fully described by the structure constants of the underlying Lie algebra. These constants satisfy algebraic constraints coming from the Jacobi identity. Similarly, every curvature-homogeneous manifold is associated with a collection of constants and field variables that satisfy algebraic and differential constraints imposed by the NP equations (2nd structure equations) and Bianchi identities.

Let  $x^i$  be a system of local coordinates on an  $n$ -dimensional manifold  $M$ . Let  $\eta_{ab} = \eta_{ba}$  be a constant inner product of a fixed signature on an  $n$ -dimensional vector space  $V \cong \mathbb{R}^n$ . We are interested in the case of  $n = 4$  and of Lorentzian signature, but much of the underlying theory can be given without these assumptions. Henceforth,  $i, j = 1, \dots, n$  are coordinate indices and  $a, b, c = 1, \dots, n$  are frame indices. We use  $\eta_{ab}$  to lower and raise frame indices as needed. Complex conjugation will be denoted by an asterisk superscript.

Let  $e_a$  be a tetrad/frame, and  $\omega^a$  be the dual coframe on  $M$ . Let  $y^a_i$  denote the covariant frame components. Thus,

$$\partial_i = \frac{\partial}{\partial x^i} = y^a_i e_a, \quad (4)$$

$$\omega^a = y^a_i dx^i, \quad (5)$$

$$g_{ij} = y^a_i y^b_j \eta_{ab}. \quad (6)$$

Let  $K^a_{bc} = -K^a_{cb}$  denote the commutator quantities (structure functions):

$$[e_b, e_c] = K^a_{bc} e_a, \quad (7)$$

$$d\omega^a = -\frac{1}{2} K^a_{bc} \omega^b \wedge \omega^c, \quad (8)$$

$$y^a_{[i,j]} = \frac{1}{2} K^a_{bc} y^b_i y^c_j. \quad (9)$$

Let

$$G = \mathcal{O}(\eta) = \{X^a_b : X_{ca} X^c_b = \eta_{ab}\}, \quad (10)$$

$$\mathfrak{g} = \mathfrak{o}(\eta) = \{A^a_b : A_{(ab)} = 0\} \quad (11)$$

denote, respectively, the

$$N := n(n-1)/2$$

dimensional group of  $\eta$ -orthogonal transformations and the corresponding Lie algebra of skew-symmetric infinitesimal transformations. Let

$$\mathbf{A}_\alpha = (A^a_{b\alpha}), \quad \Theta^\alpha = (\Theta_a^{b\alpha}),$$

be a basis of  $\mathfrak{g}$  and the dual basis, respectively, and let  $C^\alpha_{\beta\gamma}$  be the corresponding  $\mathfrak{g}$ -structure constants. Thus,

$$[\mathbf{A}_\alpha, \mathbf{A}_\beta] = C^\gamma_{\alpha\beta} \mathbf{A}_\gamma, \quad (12)$$

$$A^a_{c\alpha} A^c_{b\beta} - A^a_{c\beta} A^c_{b\alpha} = C^\gamma_{\alpha\beta} A^a_{b\gamma}, \quad (13)$$

$$A_{(ab)\alpha} = 0, \quad \Theta^{(ab)\alpha} = 0. \quad (14)$$

Henceforth,  $\alpha, \beta, \gamma = 1, \dots, N$  denote  $\mathfrak{g}$ -indices (in effect, these are bivector indices.) Let  $\Gamma^\alpha_a$  denote the connection scalars projected onto this basis. These are linearly equivalent to the commutator quantities:

$$K^a_{bc} = \Gamma^\alpha_{[c} A^a_{b]\alpha}, \quad (15)$$

$$\Gamma^\alpha_c = \Theta^{ab\alpha} (2K_{[ab]c} - K_{cab}). \quad (16)$$

Respectively, let

$$\mathbf{\Gamma}^\alpha = \Gamma^\alpha_c \omega^c, \quad (17)$$

$$\mathbf{\Omega}^\alpha = d\mathbf{\Gamma}^\alpha + \frac{1}{2} C^\alpha_{\beta\gamma} \mathbf{\Gamma}^\beta \wedge \mathbf{\Gamma}^\gamma = \frac{1}{2} R^\alpha_{bc} \omega^b \wedge \omega^c, \quad (18)$$

be the  $\mathfrak{g}$ -valued connection 1-form and the curvature 2-form. The commutator equation (8) can now be rewritten as

$$d\omega^a = -A^a_{b\alpha}\Gamma^\alpha \wedge \omega^b. \quad (19)$$

Above,

$$R^\alpha_{ab} = 2\Gamma^\alpha_{[b,a]} + C^\alpha_{\beta\gamma}\Gamma^\beta_{[a}\Gamma^\gamma_{b]} - 2\Gamma^\alpha_c\Gamma^\beta_{[a}A^c_{b]\beta} \quad (20)$$

$$= 2\Gamma^\alpha_{[b,a]} - C^\alpha_{\beta\gamma}\Gamma^\beta_{[a}\Gamma^\gamma_{b]} - 2(\mathbf{A}_\beta \cdot \Gamma)^\alpha_{[a}\Gamma^\beta_{b]}, \quad (21)$$

denote the curvature scalars, and

$$(\mathbf{A}_\beta \cdot \Gamma)^\alpha_a = C^\alpha_{\beta\gamma}\Gamma^\gamma_a - \Gamma^\alpha_c A^c_{a\beta} \quad (22)$$

denotes the action of  $\mathfrak{g}$  on  $\text{Lin}(V, \mathfrak{g})$ . The curvature scalars obey the algebraic and differential Bianchi identities; respectively,

$$R^\alpha_{[bc}A^a_{d]\alpha} = 0, \quad (23)$$

$$R^\alpha_{[ab,c]} = -(\mathbf{A}_\beta \cdot R)^\alpha_{[ab}\Gamma^\beta_{c]}, \quad (24)$$

where

$$(\mathbf{A}_\beta \cdot R)^\alpha_{ab} = C^\alpha_{\beta\gamma}R^\gamma_{ab} + 2R^\alpha_{c[a}A^c_{b]\beta} \quad (25)$$

denotes the  $\mathfrak{g}$ -action on  $\text{Lin}(\Lambda^2 V, \mathfrak{g})$ .

**2.2. The CH data.** By definition, a  $\text{CH}_k$  manifold is specified by an array of constants

$$\tilde{R}^{(k)} = (\tilde{R}^0, \tilde{R}^1, \dots, \tilde{R}^k) = (\tilde{R}^\alpha_{ab}, \tilde{R}^\alpha_{abc}, \dots, \tilde{R}^\alpha_{abc_1 \dots c_k})$$

such that

$$\nabla_{c_1 \dots c_i} \tilde{R}^\alpha_{ab} = \tilde{R}^\alpha_{abc_1 \dots c_i}, \quad i = 0, 1, \dots, k, \quad (26)$$

relative to some  $\eta$ -orthogonal frame. Note: henceforth a tilde decoration denotes an array of constants.

There are two important observations to be made at this point. First, it is more efficient to represent the algebraic CH data in terms of connection scalars rather than curvature scalars. To that end, set

$$G_{-1} := G, \quad \mathfrak{g}_{-1} := \mathfrak{g}, \quad (27)$$

and let  $G_i \subset G_{i-1}$ ,  $i = 0, 1, \dots, k$  denote the subgroup that leaves invariant

$$\tilde{R}^{(i)} := (\tilde{R}^0, \tilde{R}^1, \dots, \tilde{R}^i).$$

Let  $\mathfrak{g}_i$  denote the corresponding Lie algebra, and set

$$N_i := \dim \mathfrak{g}_i, \quad \hat{N}_i := N - N_i, \quad i = 0, \dots, k+1. \quad (28)$$

Arrange the basis of  $\mathfrak{g}$  into  $k+2$  groups of generators,

$$\mathbf{A}_{\rho_1}, \dots, \mathbf{A}_{\rho_k}, \mathbf{A}_\lambda, \mathbf{A}_\xi, \quad (29)$$

where

$$\mathbf{A}_\xi, \quad \hat{N}_k + 1 \leq \xi \leq N \quad (30)$$

is a basis of  $\mathfrak{g}_k$ , where

$$\mathbf{A}_\lambda, \mathbf{A}_\xi, \quad \hat{N}_{k-1} + 1 \leq \lambda \leq \hat{N}_k \quad (31)$$

is a basis of  $\mathfrak{g}_{k-1}$  and where

$$\mathbf{A}_{\rho_i}, \dots, \mathbf{A}_{\rho_k}, \mathbf{A}_\lambda, \mathbf{A}_\xi, \quad \hat{N}_{i-2} + 1 \leq \rho_i \leq \hat{N}_{i-1} \quad (32)$$

is a basis of  $\mathfrak{g}_{i-1}$ ,  $i = 1, \dots, k$ . Henceforth, we restrict the indices  $\xi, \lambda, \rho_i$  to the ranges indicated above, and use the Einstein convention to sum over these indices.

By the usual formula for the covariant derivative,

$$\nabla_c R^\alpha_{ab} \omega^c = dR^\alpha_{ab} + (A_\beta \cdot R)^\alpha_{ab} \Gamma^\beta, \quad (33)$$

where the second term on the right is defined in (25). In a  $\text{CH}_1$  geometry,  $R^\alpha_{ab} = \tilde{R}^\alpha_{ab}$  is constant, and since  $\mathfrak{g}_0$  leaves invariant the latter array, we actually have

$$\tilde{R}^\alpha_{abc} = (A_{\rho_1} \cdot \tilde{R})^\alpha_{ab} \Gamma^{\rho_1}_c. \quad (34)$$

The scalars  $\Gamma^{\rho_1}_a$  specify an element of  $\text{Lin}(V, \mathfrak{g}/\mathfrak{g}_0)$ . By definition of  $\mathfrak{g}_0$ , the linear map  $(\Gamma^{\rho_1}_a) \mapsto (\tilde{R}^\alpha_{abc})$  has a trivial kernel. Hence, it is possible to solve the linear system (34) and express  $\Gamma^{\rho_1}_a$  in terms of  $\tilde{R}^\alpha_{ab}$  and  $\tilde{R}^\alpha_{abc}$  — rational in the former, and linear in the latter. Therefore, in a  $\text{CH}_1$  context  $\Gamma^{\rho_1}_a = \tilde{\Gamma}^{\rho_1}_a$  is an array of constants. The following Proposition makes this more precise.

**Proposition 2.1.** *Let  $\mathfrak{g}$  be a finite-dimensional Lie algebra,  $V$  a  $\mathfrak{g}$ -module, and  $T$  the tensor algebra over  $V$ . Let us define a bilinear product on  $\mathfrak{g} \otimes T$  by setting*

$$(a \otimes \alpha) \cdot (b \otimes \beta) := [a, b] \otimes \beta \otimes \alpha + b \otimes (a \cdot \beta) \otimes \alpha, \quad a, b \in \mathfrak{g}, \quad \alpha, \beta \in T. \quad (35)$$

*This product satisfies the Leibniz rule with respect to the action of  $\mathfrak{g}$  on  $\mathfrak{g} \otimes T$ . In other words, for  $a, b, c \in \mathfrak{g}$  and  $\beta, \gamma \in T$ , we have*

$$a \cdot ((b \otimes \beta) \cdot (c \otimes \gamma)) = ([a, b] \otimes \beta + b \otimes (a \cdot \beta)) \cdot (c \otimes \gamma) + (b \otimes \beta) \cdot ([a, c] \otimes \gamma + c \otimes (a \cdot \gamma)). \quad (36)$$

*Proof.* The left-hand side of (36) expands to

$$\begin{aligned} \text{LHS} &= [a, [b, c]] \otimes \gamma \otimes \beta + [b, c] \otimes (a \cdot \gamma) \otimes \beta + [b, c] \otimes \gamma \otimes (a \cdot \beta) + \\ &\quad + [a, c] \otimes (b \cdot \gamma) \otimes \beta + c \otimes (a \cdot b \cdot \gamma) \otimes \beta + c \otimes (b \cdot \gamma) \otimes (a \cdot \beta) \\ &= [[a, b], c] \otimes \gamma \otimes \beta + c \otimes ([a, b] \cdot \gamma) \otimes \beta + [b, c] \otimes \gamma \otimes (a \cdot \beta) + c \otimes (b \cdot \gamma) \otimes (a \cdot \beta) + \\ &\quad + [b, [a, c]] \otimes \gamma \otimes \beta + [b, c] \otimes (a \cdot \gamma) \otimes \beta + [a, c] \otimes (b \cdot \gamma) \otimes \beta + c \otimes (b \cdot a \cdot \gamma) \otimes \beta. \end{aligned}$$

By inspection, the latter is equal to the right-hand side of (36).  $\square$

Henceforth, let us set

$$\tilde{\Gamma}^{(1)} := (\tilde{\Gamma}^{\rho_1}_a)$$

and use (35) to rewrite (34) as

$$\tilde{R}^1 = \tilde{\Gamma}^{(1)} \cdot \tilde{R}^0.$$

Let  $A \in \mathfrak{g}_0$ ; i.e.  $A \cdot \tilde{R}^0 = 0$ . By Proposition 2.1,

$$A \cdot \tilde{R}^1 = (A \cdot \tilde{\Gamma}^{(1)}) \cdot \tilde{R}^0 + \tilde{\Gamma}^{(1)} \cdot (A \cdot \tilde{R}^0) = (A \cdot \tilde{\Gamma}^{(1)}) \cdot \tilde{R}^0. \quad (37)$$

The above identity also establishes that  $A \in \mathfrak{g}_1$  if and only if  $A \cdot \tilde{\Gamma}^{(1)} = 0$ . Hence,  $\mathfrak{g}_1$  can be characterized as the automorphism subalgebra of  $\tilde{\Gamma}^{(1)}$ . Therefore, a  $\text{CH}_1$  geometry is fully described by the constants  $\tilde{R}^0, \tilde{\Gamma}^{(1)}$ .

In a  $\text{CH}_k$  context, formula (34) extends to covariant derivatives of higher order:

$$\tilde{R}^\alpha_{abc_1 \dots c_i} = \sum_{j=1}^i (A_{\rho_j} \cdot \tilde{R})^\alpha_{abc_1 \dots c_{i-1}} \tilde{\Gamma}^{\rho_j}_{c_i}, \quad i = 1, \dots, k. \quad (38)$$

Setting

$$\tilde{\Gamma}^{(i)} = \begin{pmatrix} \tilde{\Gamma}^{\rho_1}_a \\ \vdots \\ \tilde{\Gamma}^{\rho_i}_a \end{pmatrix} \quad (39)$$

the equation (38) can be expressed, symbolically, as

$$\tilde{R}^i = \tilde{\Gamma}^{(i)} \cdot \tilde{R}^{i-1}, \quad i = 1, \dots, k. \quad (40)$$

Therefore, a  $\text{CH}_k$  geometry is fully described by the constants  $\tilde{R}^0, \tilde{\Gamma}^{(k)}$ .

**2.3. Proper CH geometry.** The second crucial observation is that the condition that distinguishes proper CH geometries from homogeneous geometries can be restated in terms of the automorphism subalgebras  $\mathfrak{g}_i$ , c.f. Theorem 1.2.

**Theorem 2.2** (Singer's criterion, restated). *If  $M$  is a **proper**  $\text{CH}_k$  manifold, then, necessarily,  $\mathfrak{g}_k \subsetneq \dots \subsetneq \mathfrak{g}_0 \subsetneq \mathfrak{g}_{-1}$  is a chain of proper inclusions.*

This follows from Karlhede's algorithm. In a CH manifold, if  $\mathfrak{g}_i = \mathfrak{g}_{i-1}$ , then the algorithm terminates because  $t_i = t_{i-1} = 0$ . Since all differential invariants are constants, the manifold is a homogeneous space. The constants  $\tilde{R}^{(k)}$  are invariants that define the structure constants of the corresponding Lie algebra of Killing vectors. For more details, see [13] and Chapter 2.6 of [14].

Hence, if the geometry is properly  $\text{CH}_k$ , i.e., if it is not  $\text{CH}_{k+1}$ , then, at the  $(k+1)$ st order, we have

$$\nabla_{c_{k+1} \dots c_1} R^\alpha_{ab} = (\mathbf{A}_\lambda \cdot \tilde{R})^\alpha_{abc_1 \dots c_k} \Gamma^\lambda_{c_{k+1}} + \sum_{i=1}^k (\mathbf{A}_{\rho_i} \cdot \tilde{R})^\alpha_{abc_1 \dots c_k} \tilde{\Gamma}^{\rho_i}_{c_{k+1}}, \quad (41)$$

where not all  $\Gamma^\lambda_a$  are constants. Symbolically, we will express this as

$$R^{k+1} = \Gamma^{(k+1)} \cdot \tilde{R}^k,$$

where

$$\Gamma^{(k+1)} := \begin{pmatrix} \tilde{\Gamma}^{(k)} \\ \Gamma^\lambda_a \end{pmatrix}.$$

**2.4. Transformations of the CH data.** A  $\text{CH}_k$  metric does not determine the groups  $G_i$  and the constants  $\tilde{R}^0, \tilde{\Gamma}^{(k)}$  uniquely, but only up to a certain transformation. The general transformation law  $\Gamma^\alpha_a \mapsto \hat{\Gamma}^\alpha_a$  for connection scalars involves derivatives:

$$\hat{\Gamma}^\alpha_a \omega^a = (X \cdot \Gamma)^\alpha_a \omega^a + (X^{-1} dX)^\alpha. \quad (42)$$

Here,  $X$  is a  $G$ -valued function on  $M$ , and  $X \cdot \Gamma$  denotes the  $G$ -action on  $\text{Lin}(V, \mathfrak{g})$ . Note that  $(\Gamma^\lambda_a)$  is a field taking values in  $\text{Lin}(V, \mathfrak{g}_{k-1}/\mathfrak{g}_k)$ . Hence, if we restrict the values of the frame transformation to  $G_k$ , i.e.,  $X : M \rightarrow G_k$ , then  $X^{-1} dX$  takes values in  $\mathfrak{g}_k$ , and the transformation law for the connection components modulo  $\mathfrak{g}_k$  becomes tensorial:

$$\hat{\Gamma}^\lambda_a = (X \cdot \Gamma)^\lambda_a. \quad (43)$$

This makes sense, because the scalars  $\Gamma^\lambda_a$  depend linearly on  $R^{k+1}$ , and the components of the latter transform tensorially.

Thus, in a  $\text{CH}_k$  manifold the group  $G_0$  is only determined up to an  $G_{-1}$  conjugation. If  $X \in G_{-1}$  is a constant frame transformation, i.e., if  $dX = 0$ , then the corresponding frame transformation leaves  $\tilde{R}^0, \tilde{\Gamma}^{(k)}$  constant. If  $G_0$  is fixed,

then the constants  $\tilde{R}^0$  are determined up to a choice of conjugation by a constant  $X \in \mathbf{N}(G_0)$ , where the latter denotes the normalizer of  $G_0$ . More generally, once  $G_i$ ,  $i = 1, \dots, k-1$  is fixed, then  $G_{i+1}$  and the constants  $\tilde{\Gamma}^{(i+1)}$  are determined up to conjugation by a constant frame transformation  $X \in \mathbf{N}(G_0) \cap \dots \cap \mathbf{N}(G_i)$ . The latter is the group that preserves the chain  $G_i \subset \dots \subset G_0 \subset G_{-1}$ . Once  $G_k$  has been fixed, the constant data is fixed. However, the scalars  $\Gamma^\lambda_a$  obey a  $G_k$  transformation law (43), and can be normalized using a non-constant frame transformation  $X : M \rightarrow G_k$ .

**2.5. The CH constraints.** The  $\text{CH}_k$  condition imposes certain algebraic and differential constraints on the constants  $\tilde{R}^0 = (\tilde{R}^\alpha_{ab})$ ,  $\tilde{\Gamma}^{(k)} = (\tilde{\Gamma}^{\rho_i}_a)_{i=1}^k$  and field variables  $\Gamma^\lambda_a$ . To express these, we introduce the following quantities:

$$\tilde{\xi}^a_{bcd} := \tilde{R}^\alpha_{[bc} A^a_{d]\alpha} \quad (44)$$

$$\tilde{\Xi}^\alpha_{abc} := (\mathbf{A}_{\rho_1} \cdot \tilde{R})^\alpha_{[ab} \tilde{\Gamma}^{\rho_1}_{c]} \quad (45)$$

$$\begin{aligned} \tilde{\Upsilon}^{\rho_i}_{ab} &:= \tilde{R}^{\rho_i}_{ab} - \sum_{\sigma, \tau=1}^{\tilde{N}_{i-1}} C^{\rho_i}_{\sigma\tau} \tilde{\Gamma}^\sigma_a \tilde{\Gamma}^\tau_b + 2 \sum_{\sigma=1}^{\tilde{N}_{i-1}} \tilde{\Gamma}^{\rho_i}_c \tilde{\Gamma}^\sigma_{[a} A^c_{b]\sigma} + \\ &\quad + 2(\mathbf{A}_{\rho_{i+1}} \cdot \tilde{\Gamma})^{\rho_i}_{[a} \tilde{\Gamma}^{\rho_{i+1}}_{b]}, \quad i = 1, \dots, k-1; \end{aligned} \quad (46)$$

$$\begin{aligned} \Upsilon^{\rho_k}_{ab} &:= \tilde{R}^{\rho_k}_{ab} - \sum_{\sigma, \tau=1}^{\tilde{N}_{k-1}} C^{\rho_k}_{\sigma\tau} \tilde{\Gamma}^\sigma_a \tilde{\Gamma}^\tau_b + 2 \sum_{\sigma=1}^{\tilde{N}_{k-1}} \tilde{\Gamma}^{\rho_k}_c \tilde{\Gamma}^\sigma_{[a} A^c_{b]\sigma} + \\ &\quad + 2(\mathbf{A}_\lambda \cdot \tilde{\Gamma})^{\rho_k}_{[a} \Gamma^\lambda_{b]}; \end{aligned} \quad (47)$$

$$\begin{aligned} \Upsilon^\lambda_{ab} &:= \tilde{R}^\lambda_{ab} - \sum_{\sigma, \tau=1}^{\tilde{N}_{k-1}} C^\lambda_{\sigma\tau} \tilde{\Gamma}^\sigma_a \tilde{\Gamma}^\tau_b + 2 \sum_{\sigma=1}^{\tilde{N}_{k-1}} (\Gamma^\lambda_c \tilde{\Gamma}^\sigma_{[a} A^c_{b]\sigma} - \Gamma^\mu_a \tilde{\Gamma}^\sigma_b C^\lambda_{\mu\sigma}) \\ &\quad - C^\lambda_{\mu\nu} \Gamma^\mu_a \Gamma^\nu_b + 2 \Gamma^\lambda_c \Gamma^\mu_{[a} A^c_{b]\mu}, \end{aligned} \quad (48)$$

where in (48)  $\mu, \nu$  have the same range as  $\lambda$ , as per (31). The algebraic and differential Bianchi identities (23) (24) reduce to the following constraints:

$$\tilde{\xi}^a_{bcd} = 0, \quad (49)$$

$$\tilde{\Xi}^\alpha_{bcd} = 0. \quad (50)$$

The NP equations (20) corresponding to generators  $\mathbf{A}_{\rho_1}, \dots, \mathbf{A}_{\rho_k}$  also reduce to the following algebraic constraints

$$\tilde{\Upsilon}^{\rho_i}_{ab} = 0, \quad i = 1, \dots, k-1, \quad (51)$$

$$\Upsilon^{\rho_k}_{ab} = 0. \quad (52)$$

The NP equations corresponding to generators  $\mathbf{A}_\lambda$  give  $\text{CH}_k$  field equations:

$$d\Gamma^\lambda_b \wedge \omega^b + (\mathbf{A}_\xi \cdot \Gamma)^\lambda_b \Gamma^\xi_a \wedge \omega^a = \frac{1}{2} \Upsilon^\lambda_{ab} \omega^a \wedge \omega^b, \quad (53)$$

$$\Gamma^\lambda_{[a,b]} = (\mathbf{A}_\xi \cdot \Gamma)^\lambda_{[a} \Gamma^\xi_{b]} - \frac{1}{2} \Upsilon^\lambda_{ab}. \quad (54)$$

The field equations for the scalars  $\Gamma^\xi_a$  are similar. Note that if  $G_k$  is trivial, as happens in the  $\text{CH}_2$  examples derived in Section 3, then (54) becomes, simply

$$\Gamma^\lambda_{[a,b]} = -\frac{1}{2} \Upsilon^\lambda_{ab}. \quad (55)$$

**2.6. Integrability conditions.** Equations (51) are polynomial constraints on the constants  $\tilde{R}^0, \tilde{\Gamma}^{(k)}$ . Equations (52) are linear algebraic constraints while (54) are quasi-linear differential equations in the field variables  $\Gamma^\lambda_a$ . The scalars  $\Gamma^\xi_a$  are not subject to any algebraic constraints. The equations (51) (52) (54) are necessary conditions, but not, in general, sufficient to describe a  $\text{CH}_k$  geometry because of the presence of integrability conditions. The complication, roughly speaking, is that the derivatives of the algebraic constraints (52) together with differential constraints (54) may imply additional zero-order (algebraic) constraints (obtained by eliminating all first-order terms.) The derivatives of these zero-order constraints may imply further algebraic constraints, etc. In addition to zero-order integrability conditions, equations (54) may fail to be involutive because of first order obstructions. Taking derivatives of (54) yields second-order differential equations in  $\Gamma^\lambda_a$ . It is conceivable that a particular linear combination of these prolonged second-order equations eliminates all second-order derivatives, and furnishes additional first-order constraints that are independent of (54).

Fix  $\eta_{ab}$  of the desired signature. Henceforth,  $x^i, y^a_i, \Gamma^\alpha_a$  are canonical coordinates. Let  $\text{FM}$  denote the  $\text{GL}_n\mathbb{R}$  frame bundle over  $M$ . The variables  $x^i, y^a_i$  are canonical coordinates on  $\text{FM}$ , while the variables  $\Gamma^\alpha_a$  are canonical coordinates on the vector space  $\text{Lin}(V, \mathfrak{g})$ . The  $x^i$  are independent variables, while

$$y^a_i = y^a_i(x^1, \dots, x^n), \quad (56)$$

$$\Gamma^\alpha_i = \Gamma^\alpha_i(x^1, \dots, x^n) \quad (57)$$

are the dependent variables. The metric  $g_{ij}$ , as given by (6), is a derived dependent variable. The differential forms  $\omega^a, \Gamma^\alpha, \Omega^\alpha$  defined, respectively, by (5)(17) (18) should also be regarded as unknown quantities.

Let

$$\tilde{R}^0 = (\tilde{R}^\alpha_{ab}) \in \text{Lin}(\Lambda^2 V, \mathfrak{g}), \quad (58)$$

$$\tilde{\Gamma}^{(j)} = (\tilde{\Gamma}^{\rho_i}_a)_{i=1}^j \in \text{Lin}(V, \mathfrak{g}_{j-2}/\mathfrak{g}_{j-1}), \quad j = 1, \dots, k \quad (59)$$

be arrays of constants that satisfy (49) (50) (51). The first set of constraints comes from algebraic Bianchi relations, the second set from differential Bianchi, and the third set from tier 1 through  $k-1$  NP equations. Let us again emphasize that in a  $\text{CH}_k$  context some differential constraints reduce to purely algebraic constant constraints. In (59), the automorphism Lie algebras are defined inductively by

$$\mathfrak{g}_{-1} = \mathfrak{g}, \quad (60)$$

$$\mathfrak{g}_0 = \text{Aut } \tilde{R}^0 = \{A \in \mathfrak{g}_{-1} : (A \cdot \tilde{R})^\alpha_{ab} = 0\}, \quad (61)$$

$$\mathfrak{g}_i = \text{Aut } \tilde{R}^{(i)} \quad (62)$$

$$\begin{aligned} &= \text{Aut } \tilde{R}^0 \cap \text{Aut } \tilde{R}^1 \cap \dots \cap \text{Aut } \tilde{R}^i \\ &= \text{Aut } \tilde{R}^0 \cap \text{Aut } \Gamma^{(i)} \\ &= \text{Aut}\{A \in \mathfrak{g}_{i-1} : (A \cdot \tilde{\Gamma})^{\rho_i}_a = 0\}, \quad i = 1, \dots, k \end{aligned}$$

Let  $Z \subset \text{Lin}(V, \mathfrak{g})$  be an algebraic variety defined by the equations

$$\Gamma^{\rho_i}_a = \tilde{\Gamma}^{\rho_i}_a, \quad i = 1, \dots, k, \quad (63)$$

by the linear equations (52), and by some additional polynomial equations in the  $\Gamma^\lambda_a$ .

**Definition 2.3.** We will call the pair  $(\tilde{R}^0, Z)$  a  $\text{CH}_k$  configuration. We will say that the configuration is proper, if every inclusion  $\mathfrak{g}_i \subsetneq \mathfrak{g}_{i-1}$  is proper. We will say that two configurations are equivalent if they can be related by a constant  $\mathcal{O}(\eta)$  conjugation.

Let  $ZM = FM \times Z$ . Set

$$\tau^a := d\omega^a - A^a_{b\alpha}\Gamma^\alpha, \quad (64)$$

$$\Upsilon^\alpha := \Omega^\alpha - \frac{1}{2}\tilde{R}^\alpha_{ab}\omega^a \wedge \omega^b, \quad (65)$$

and let  $(\mathcal{J}, \Theta)$  be the exterior differential system[26] [16, Chapter 18] on  $ZM$  generated by the 2-forms  $\tau^a, \Omega^\lambda, \Omega^\xi$ , subject to the independence condition

$$\Theta = dx^1 \wedge \cdots \wedge dx^n \neq 0. \quad (66)$$

Equivalently, as per (9) (15), we may consider scalar equations

$$y^a_{[i,j]} = \frac{1}{2}\Gamma^\alpha_{[cA^a]_b]\alpha}, \quad (67)$$

and NP equations (54) constrained by the variety  $Z$ .

**Definition 2.4.** We will say that a  $\text{CH}_k$  configuration is free of torsion if all zero order integrability constraints are satisfied identically on  $ZM$ , i.e., if there exists an  $n$ -dimensional integral element of  $(\mathcal{J}, \Theta)$  above every point of  $ZM$ .

**Proposition 2.5.** The exterior ideal  $\mathcal{J}$  is differentially closed.

*Proof.* By (18),

$$\Upsilon^\alpha = d\Gamma^\alpha + \frac{1}{2}C^\alpha_{\beta\gamma}\Gamma^\beta \wedge \Gamma^\gamma - \frac{1}{2}\tilde{R}^\alpha_{ab}\omega^a \wedge \omega^b$$

Since  $\Gamma^{(k)}$  are constants, we have

$$\Upsilon^{\rho_i} = \frac{1}{2}\Upsilon^{\rho_i}_{ab}\omega^a \wedge \omega^b = 0, \quad i = 1, \dots, k$$

by definition. Therefore,  $\Upsilon^\alpha \in \mathcal{J}$  for all  $\alpha$ . As well, the following identities hold:

$$d\tau^a \equiv \frac{1}{6}\xi^a_{bcd}\omega^a \wedge \omega^b \wedge \omega^c \pmod{\mathcal{J}}, \quad (68)$$

$$d\Upsilon^\alpha \equiv \frac{1}{6}\Xi^\alpha_{abc}\omega^a \wedge \omega^b \wedge \omega^c \pmod{\mathcal{J}}, \quad (69)$$

with  $\xi^a_{bcd}, \Xi^\alpha_{bcd}$  defined in (44) (45). Again, by definition, these polynomials vanish on  $Z$ . Therefore, the three-forms  $d\tau^a, d\Upsilon^\alpha$  all belong to  $\mathcal{J}_{\mathcal{R}}$ .  $\square$

In and of itself, the above result does not guarantee involutivity because of the potential presence of zero and first order integrability constraints. However, in light of the above result, the construction of proper  $\text{CH}_k$  geometries is reduced to the search for proper, torsion-free configurations up to  $G$ -equivalence. After classifying proper, torsion-free  $\text{CH}_k$  configurations, all that remains is to test these configurations for involutivity, that is the absence of additional 1st-order integrability conditions. Below, we apply this approach to classify all proper  $\text{CH}_2$  Lorentzian four-manifolds, and to prove the non-existence of proper  $\text{CH}_3$  Lorentzian four-manifolds.

The algebraic constraints (52) and the differential constraints (54) are both consequences of the NP equations. An analysis of the branching arising from the

zero-order integrability conditions implied by these constraints leads to a classification of proper, torsion-free configurations. We implement this program for the case of four-dimensional Lorentzian  $\text{CH}_2$  geometries in the next section.

### 3. PROPER $\text{CH}_2$ GEOMETRIES

**3.1. Derivation of proper configurations.** In this section we classify proper  $\text{CH}_2$  Lorentzian geometries. Our method relies in an essential way on Theorem 2.2, Singer's criterion. We begin by classifying proper Lie algebra chains  $\mathfrak{g}_2 \subsetneq \mathfrak{g}_1 \subsetneq \mathfrak{g}_0 \subsetneq \mathfrak{g}_{-1}$ , where  $\mathfrak{g}_{-1}$  is the 6-dimensional Lie algebra of infinitesimal Lorentz transformations, where  $\mathfrak{g}_0$  is the automorphism algebra for some curvature constants  $\tilde{R}^0 = (\tilde{R}^\alpha_{ab})$ , and where  $\mathfrak{g}_i$ ,  $i = 1, 2$  is the automorphism algebra of some tier  $i$  connection constants  $\tilde{\Gamma}^{(i)}$ . By focusing on proper chains, we exclude homogeneous four-dimensional geometries. These are classified, in the Riemannian case, in [27], and in the indefinite signature case in [28].

Henceforth, we assume that  $M$  is a four-dimensional, analytic manifold and that  $\eta_{ab}$  is the Lorentzian inner product. We will express our calculations using the Newman-Penrose (NP) formalism [8, Chapter 2], which is based on complexified null tetrads

$$(e_a) = (\mathbf{m}, \bar{\mathbf{m}}, \mathbf{n}, \ell) = (\delta, \delta^*, \Delta, D).$$

The inner product and the metric are given by

$$g_{ij} dx^i dx^j = \eta_{ab} \omega^a \omega^b = 2\omega^1 \omega^2 - 2\omega^3 \omega^4, \quad (70)$$

The connection scalars  $\Gamma^\alpha_a$  are labeled by the 12 complex-valued NP spin coefficients:

$$\begin{aligned} -\omega_{14} &= \sigma\omega^1 + \rho\omega^2 + \tau\omega^3 + \kappa\omega^4; \\ \omega_{23} &= \mu\omega^1 + \lambda\omega^2 + \nu\omega^3 + \pi\omega^4; \\ -\frac{1}{2}(\omega_{12} + \omega_{34}) &= \beta\omega^1 + \alpha\omega^2 + \gamma\omega^3 + \epsilon\omega^4. \end{aligned}$$

The curvature scalars are labeled by the Ricci scalar  $\Lambda = \bar{\Lambda}$ , Hermitian Ricci components  $\Phi_{AB} = \bar{\Phi}_{BA}$ ,  $A, B = 0, 1, 2$ , and complex Weyl components  $\Psi_C$ ,  $C = 0 \dots 4$  according to the usual Newman-Penrose scheme:

$$\Omega_{14} = \Phi_{01}(\omega^3 \wedge \omega^4 - \omega^1 \wedge \omega^2) + \Psi_1(\omega^1 \wedge \omega^2 + \omega^3 \wedge \omega^4) - \Phi_{02}\omega^1 \wedge \omega^3 + \Phi_{00}\omega^2 \wedge \omega^4 + \Psi_0\omega^1 \wedge \omega^4 - (\Psi_2 + 2\Lambda)\omega^2 \wedge \omega^3 \quad (71)$$

$$\Omega_{23} = \Phi_{21}(\omega^1 \wedge \omega^2 - \omega^3 \wedge \omega^4) - \Psi_3(\omega^1 \wedge \omega^2 + \omega^3 \wedge \omega^4) + \Phi_{22}\omega^1 \wedge \omega^3 - \Phi_{20}\omega^2 \wedge \omega^4 + \Psi_4\omega^2 \wedge \omega^3 - (\Psi_2 + 2\Lambda)\omega^1 \wedge \omega^4 \quad (72)$$

$$\begin{aligned} \frac{1}{2}(\Omega_{12} + \Omega_{34}) &= \Phi_{11}(\omega^3 \wedge \omega^4 - \omega^1 \wedge \omega^2) + (\Psi_2 - \Lambda)(\omega^1 \wedge \omega^2 + \omega^3 \wedge \omega^4) \\ &\quad - \Phi_{12}\omega^1 \wedge \omega^3 + \Phi_{10}\omega^2 \wedge \omega^4 + \Psi_1\omega^1 \wedge \omega^4 - \Psi_3\omega^2 \wedge \omega^3 \end{aligned} \quad (73)$$

If the Petrov type is I, II, or III, then one can fully fix the frame by setting  $\Psi_0 = \Psi_4 = 0$  and then normalizing  $\Psi_1$  or  $\Psi_3$  to 1. In other words,  $N_1 = N_0 = 0$ , and hence, by Singer's criterion, every  $\text{CH}_1$  manifold of Petrov type I, II, or III is a homogeneous space. Proper  $\text{CH}_1$  Lorentzian manifolds must, necessarily, be of Petrov type D, N, or O.

A proper  $\text{CH}_2$  configuration, if one exists, requires that  $N_0 \geq 2$ . Modulo conjugation by a Lorentz transformation, there are only 5 types of curvature tensor for which the automorphism group has dimension 2 or higher. The analysis of these 5

cases and their subcases is detailed below. Only in case 5.2, do we obtain a proper  $\text{CH}_2$  configuration.

A proper  $\text{CH}_3$  configuration, if one exists, requires a proper chain  $\mathfrak{g}_3 \subsetneq \mathfrak{g}_2 \subsetneq \mathfrak{g}_1 \subsetneq \mathfrak{g}_0 \subsetneq \mathfrak{g}_{-1}$ , and hence requires that  $N_0 = \dim \mathfrak{g}_0 \geq 3$ . Thus the search for proper  $\text{CH}_3$  configurations is limited to cases 1,2,3, below. However, for each of these cases we rule out the existence of a proper  $\text{CH}_2$  configuration. From this it follows that there does not exist a proper  $\text{CH}_3$  configuration, and hence there does not exist a proper  $\text{CH}_3$  Lorentzian four-manifold.

Case 1.  $\mathfrak{g}_0 = \mathfrak{so}(3)$ . The curvature is that of a conformally flat perfect fluid.

$$\Psi_A = \Phi_{01} = \Phi_{02} = \Phi_{12} = 0. \quad \Phi_{00} = \Phi_{22} = 2\Phi_{11} \neq 0.$$

As a basis of  $\mathfrak{o}(\eta)$  we take

$$(\mathbf{A}_\alpha) = (e^{34}, e^{14}, e^{24}, e^{13} - e^{14}, e^{23} - e^{24}, e^{12}), \quad (74)$$

where  $e^{ab} = e^a \wedge e^b$  is a basic bivector. Note that, as per the indexing scheme (32) - (30),  $\mathbf{A}_4, \mathbf{A}_5, \mathbf{A}_6$  are the  $\mathfrak{so}(3)$  generators. The dual connection components are

$$(\mathbf{\Gamma}^\alpha) = (\omega_{34}, \omega_{13} + \omega_{14}, \omega_{23} + \omega_{24}, \omega_{13}, \omega_{23}, \omega_{12}).$$

Hence, the tier-1 connection constants are

$$\tilde{\Gamma}^{(1)} = \begin{pmatrix} -\beta - \bar{\alpha} & -\alpha - \bar{\beta} & -2\gamma_1 & -2\epsilon_1 \\ -\sigma + \bar{\lambda} & -\rho + \bar{\mu} & -\tau + \bar{\nu} & -\kappa + \bar{\pi} \\ \mu - \bar{\rho} & \lambda - \bar{\sigma} & \nu - \bar{\tau} & \pi - \bar{\kappa} \end{pmatrix},$$

where the 1 and 2 subscripts indicates the real part and imaginary part; e.g.,  $\gamma = \gamma_1 + i\gamma_2$ . The Bianchi identities  $\tilde{\Xi}^\alpha_{abc} = 0$  imply that all  $\tilde{\Gamma}^{\rho_1}_a = 0$ . Hence, necessarily,  $\mathfrak{g}_1 = \mathfrak{g}_0$ . Therefore, this case does not admit a proper  $\text{CH}_1$  configuration, much less a proper  $\text{CH}_2$  configuration.

Case 2.  $\mathfrak{g}_0 = \mathfrak{so}(1, 2)$ . The curvature constants are

$$\Psi_A = \Phi_{01} = \Phi_{02} = \Phi_{12} = 0. \quad \Phi_{00} = \Phi_{22} = -2\Phi_{11} \neq 0.$$

This case is quite similar to Case 1. Again, by the Bianchi identities,  $\mathfrak{g}_1 = \mathfrak{g}_0$ . This type of curvature tensor does not admit proper  $\text{CH}_1$  configurations.

Case 3. The curvature is that of an aligned null radiation field on a conformally flat background:

$$\Psi_C = \Phi_{00} = \Phi_{01} = \Phi_{11} = \Phi_{02} = \Phi_{12} = 0, \quad \Phi_{22} \neq 0.$$

The automorphism group of the curvature tensor is three-dimensional, generated by spins and null rotations; the generators are  $\mathbf{A}_4, \mathbf{A}_5, \mathbf{A}_6$  where

$$(\mathbf{A}_\alpha) = (e^{34}, e^{14}, e^{24}, e^{12}, e^{13} + e^{23}, e^{13} - e^{23}).$$

The tier-1 connection constants are

$$(\tilde{\Gamma}^{\rho_1}_a) = \begin{pmatrix} -\beta - \bar{\alpha} & -\alpha - \bar{\beta} & -2\gamma_1 & -2\epsilon_1 \\ -\sigma & -\rho & -\tau & -\kappa \\ -\bar{\rho} & -\bar{\sigma} & -\bar{\tau} & -\bar{\kappa} \end{pmatrix}$$

By the Bianchi identities, necessarily

$$\kappa = \sigma = \rho = 0, \quad \alpha = \bar{\tau}/2 - \bar{\beta}.$$

If  $\tau = 0$ , then  $\mathfrak{g}_1 = \mathfrak{g}_0$ . By Singer's criterion, this gives a homogeneous space. So, we assume that  $\tau \neq 0$ . Conjugating by a spin (a  $G_0$  transformation), as necessary, we

assume without loss of generality that  $\tau = \tau_1$  is real, and that  $\mathfrak{g}_1$  is 1-dimensional, generated by imaginary null rotations  $\mathbf{A}_6$ . Thus,

$$(\tilde{\Gamma}^{\rho_2}_a) = \begin{pmatrix} \tau/2 - 2\beta & -\tau/2 + 2\bar{\beta} & -2i\gamma_2 & -2i\epsilon_2 \\ (\mu + \bar{\lambda})/2 & (\lambda + \bar{\mu})/2 & \nu_1 & \pi_1 \end{pmatrix},$$

The tier-1 NP equations,  $\tilde{\Upsilon}^{\rho_1}_{ab} = 0$ , imply

$$\beta = -\tau/4, \quad \pi_1 = -\tau, \quad \epsilon_2 = 0, \quad \Lambda = -\tau^2, \quad \lambda = -2\gamma/3 - \bar{\mu}.$$

The above constraints describe a proper  $\text{CH}_1$  configuration. However, the tier-2 NP equations,  $\Upsilon^{\rho_2}_{ab} = 0$ , imply

$$\Phi_{22} = 8\gamma_1^2/9 - 2\nu_1\tau, \quad \gamma_2 = 0.$$

The last condition implies that  $\mathfrak{g}_2 = \mathfrak{g}_1$ . Therefore, this case does not admit a proper  $\text{CH}_2$  configuration.

Case 4. The curvature tensor has the form below. The Petrov type is D or O, with a non-null Maxwell field.

$$\Psi_0 = \Psi_1 = \Psi_3 = \Psi_4 = \Phi_{00} = \Phi_{01} = \Phi_{02} = \Phi_{12} = \Phi_{22} = 0.$$

The automorphism group of the curvature tensor is 2-dimensional, generated by boosts and spins  $\mathbf{A}_5, \mathbf{A}_6$ , where

$$(\mathbf{A}_\alpha) = (e^{14}, e^{24}, e^{13}, e^{23}, \{e^{34}, e^{12}\}).$$

The order of the last 2 generators varies according to the 2 subcases below. The tier-1 connection constants are shown below

$$(\tilde{\Gamma}^{\rho_1}_a) = \begin{pmatrix} -\sigma & -\rho & -\tau & -\kappa \\ -\bar{\rho} & -\bar{\sigma} & -\bar{\tau} & -\bar{\kappa} \\ \bar{\lambda} & \bar{\mu} & \bar{\nu} & \bar{\pi} \\ \mu & \lambda & \nu & \pi \end{pmatrix}$$

A proper  $\text{CH}_2$  configuration requires  $N_1 = 1, N_2 = 0$ . There are 2 subcases. Our analysis shows that neither subcase admits a proper configuration.

Case 4.1.  $G_1$  is generated by spins;  $\mathbf{A}_6 = e^{12}$ . This requires

$$\kappa = \sigma = \lambda = \nu = \tau = \pi = 0, \quad (\rho, \mu) \neq (0, 0).$$

The Bianchi identities imply

$$\Psi_2 = -2\Phi_{11}/3, \quad \mu_1 = 0, \quad \rho_1 = 0.$$

From there, the tier-1 NP equations imply

$$\begin{aligned} \tilde{\Upsilon}^1_{24} &= -i\rho_2(2\epsilon_1 + i\rho_2) = 0, \\ \tilde{\Upsilon}^3_{23} &= -i\mu_2(-2\gamma_1 + i\mu_2) = 0. \end{aligned}$$

Hence,  $\rho = \mu = 0$ , a contradiction.

Case 4.2.  $G_1$  is generated by boosts;  $\mathbf{A}_6 = e^{34}$ . This requires

$$\kappa = \beta = \lambda = \nu = \rho = \mu = 0, \quad (\tau, \pi) \neq (0, 0).$$

The Bianchi identities imply

$$\Psi_2 = 2\Phi_{11}/3, \quad \pi = \bar{\tau}.$$

From there, the tier-1 NP equations imply

$$\begin{aligned} \tilde{\Upsilon}^1_{13} &= \tau(\tau - i(\bar{\alpha} - \beta)) = 0 \\ \tilde{\Upsilon}^3_{14} &= \tau(\tau + i(\bar{\alpha} + \beta)) = 0. \end{aligned}$$

Hence,  $\pi = \tau = 0$ , a contradiction.

Case 5. The curvature tensor is null radiation/vacuum with an aligned type N or conformally flat background:

$$\Psi_0 = \Psi_1 = \Psi_2 = \Psi_3 = \Phi_{00} = \Phi_{01} = \Phi_{11} = \Phi_{02} = \Phi_{12} = 0, \quad (\Phi_{22}, \Psi_4) \neq (0, 0).$$

The automorphism group  $G_0$  is 2-dimensional, generated by null rotations  $\mathbf{A}_5, \mathbf{A}_6$ , where

$$(\mathbf{A}_\alpha) = (e^{14}, e^{24}, e^{34}, e^{12}, e^{13} + e^{23}, e^{13} - e^{23}). \quad (75)$$

The tier-1 connection constants are

$$(\tilde{\Gamma}^{\rho_1}_a) = \begin{pmatrix} -\sigma & -\rho & -\tau & -\kappa \\ -\bar{\rho} & -\bar{\sigma} & -\bar{\tau} & -\bar{\kappa} \\ -\beta - \bar{\alpha} & -\alpha - \bar{\beta} & -2\gamma_1 & -2\epsilon_1 \\ -\beta + \bar{\alpha} & -\alpha + \bar{\beta} & -2i\gamma_2 & -2i\epsilon_2 \end{pmatrix}$$

To obtain a proper  $\text{CH}_2$  configuration, we require  $N_0 = 2, N_1 = 1$  and  $N_2 = 0$ . Conjugating by a spin, as necessary (the normalizer of  $G_0$  is the 4-dimensional group consisting of spins, boosts and null rotations about  $\ell$ ), without loss of generality we assume that  $G_1$  is generated by imaginary null rotations  $\mathbf{A}_6$ . This requires

$$\kappa = \sigma = \rho = \epsilon = 0, \quad \alpha = \beta + \tau \neq 0.$$

The tier-2 connection constants are

$$(\tilde{\Gamma}^{\rho_2}_a) = \begin{pmatrix} \frac{1}{2}(\mu + \bar{\lambda}) & \frac{1}{2}(\lambda + \bar{\mu}) & \nu_1 & \pi_1 \end{pmatrix}.$$

The tier-1 NP constraints,

$$\tilde{\Upsilon}^1_{13} = \tilde{\Upsilon}^1_{23} = \tilde{\Upsilon}^2_{13} = \tilde{\Upsilon}^2_{23} = \tilde{\Upsilon}^3_{12} = \tilde{\Upsilon}^4_{12} = \tilde{\Upsilon}^3_{34} = 0$$

imply

$$\tau_2 = 0, \quad \beta_2 = 0, \quad \Lambda = -\tau^2, \quad \pi_1 = -\tau_1.$$

The NP constraint

$$3\tilde{\Upsilon}^3_{13} - 3\tilde{\Upsilon}^3_{3,2} - 2\tilde{\Upsilon}^4_{13} = -4i(4\gamma_2 + 3\lambda_2 - 3\mu_2)(\beta + \tau) = 0$$

then implies

$$\lambda_2 = \mu_2 - 4\gamma_2/3.$$

The NP constraint

$$\tilde{\Upsilon}^4_{13} = 2i\gamma_2(2\beta - \tau) = 0, \quad (76)$$

means that we have to analyze two subcases: the singular case  $\gamma_2 = 0$ , and the generic case  $\gamma_2 \neq 0$ .

Case 5.1. Suppose that  $\gamma_2 = 0$ . The connection constants  $\gamma_1, \tau, \beta$  are left invariant by imaginary null rotations about  $\ell$ , and hence  $\mathfrak{g}_1$  leaves invariant  $\tilde{\Gamma}^{(1)}$ . This implies that  $\mathfrak{g}_2 = \mathfrak{g}_1$ , and therefore, the present subcase does not admit a proper CH<sub>2</sub> configuration.

Case 5.2. Suppose that  $\gamma_2 \neq 0$ . The constraint (76) implies

$$\beta = \tau/2.$$

The assumption  $\gamma_2 \neq 0$  implies  $\mathfrak{g}_2 = \{0\}$ . The field variables are

$$(\Gamma^\lambda_a) = \begin{pmatrix} \frac{1}{2}(-\mu + \bar{\lambda}) & \frac{1}{2}(-\lambda + \bar{\mu}) & -i\nu_2 & -i\pi_2 \end{pmatrix}.$$

The NP constraint

$$\tilde{\Upsilon}^3_{13} = (2\gamma_1 - 3\lambda_1 - 3\mu_1)\tau = 0,$$

implies that

$$\lambda_1 = 2\gamma_1/3 - \mu_1.$$

Finally, we apply the Bianchi identities:

$$\begin{aligned} \tilde{\Xi}^5_{123} &= \frac{1}{2}\tau(\Psi_4 - \bar{\Psi}_4) = 0 \\ \tilde{\Xi}^6_{123} &= \frac{1}{2}\tau(6\Phi_{22} - \Psi_4 - \bar{\Psi}_4) = 0 \end{aligned}$$

These imply

$$\Psi_4 = 3\Phi_{22}.$$

The normalizer of the chain  $G_1 \subset G_0 \subset G_{-1}$  is the 3-dimensional group generated by boosts and null rotations about  $\ell$ . Conjugation by a real null rotation about  $\ell$  transforms the remaining tier-1 constants according to

$$\tau \mapsto \tau, \quad \gamma_1 \mapsto \gamma_1 + 2x(\beta_1 + \tau_1), \quad \gamma_2 \mapsto \gamma_2,$$

where  $x$  is the constant, real-valued transformation parameter. A boost conjugation has the following effect:

$$\tau \mapsto \tau, \quad \gamma \mapsto a\gamma,$$

where  $a \neq 0$  is the constant, real-valued boost parameter. Hence, conjugating by a real null rotation and a boost, as necessary, without loss of generality we set

$$\gamma = 3i/2. \tag{77}$$

An imaginary null rotation transforms the remaining tier-2 constants according to

$$\nu_1 \mapsto \nu_1 - 10x\gamma_2/3$$

where  $x$  is the constant, real-valued transformation parameter. Hence, without loss of generality

$$\nu_1 = 0.$$

The tier-2 NP constraint  $\Upsilon^5_{34} = 0$  implies

$$\pi_2 = 0.$$

The NP constraint  $\Upsilon^5_{13} = 0$  implies

$$\Phi_{22} = 5\tilde{\mu}_2/2 - 4, \quad \mu = i\tilde{\mu}_2$$

where  $\tilde{\mu}_2$  is a real constant.

**3.2. Involutivity of the  $\text{CH}_2$  configuration.** Let  $\tilde{\tau}_1 \neq 0, \tilde{\mu}_2 \neq 8/5$  be real constants. As per the above derivation, up to  $\mathcal{O}(\eta)$  equivalence, the most general proper  $\text{CH}_2$  configuration is given by

$$\tau = -\pi = 2\beta = 2\alpha/3 = \tilde{\tau}_1, \quad (78)$$

$$\gamma = 3i/2, \quad (79)$$

$$\mu = \lambda + 2i = i\tilde{\mu}_2, \quad (80)$$

$$\nu_1 = 0, \quad (81)$$

$$\Phi_{22} = \Psi_4/3 = -4 + 5\tilde{\mu}_2/2, \quad (82)$$

$$\Lambda = -\tilde{\tau}_1^2, \quad (83)$$

with all other connection and curvature scalars equal to zero. Equivalently, relative to the basis (75), the above configuration may be described as follows:

$$\Gamma^1 = \Gamma^2 = -\tilde{\tau}_1 \omega^3, \quad (84)$$

$$\Gamma^3 = -2\tilde{\tau}_1 (\omega^1 + \omega^2), \quad (85)$$

$$\Gamma^4 = \tilde{\tau}_1 (\omega^1 - \omega^2) - 3i \omega^3, \quad (86)$$

$$\Gamma^5 = i(\omega^1 - \omega^2) + i\tilde{\tau}_1 \omega^4, \quad (87)$$

$$\Gamma^6 = i(1 - \tilde{\mu}_2)(\omega^1 + \omega^2) - i\nu_2 \omega^3, \quad (88)$$

$$\Omega^1 = 2\tilde{\tau}_1^2 \omega^2 \wedge \omega^3, \quad (89)$$

$$\Omega^2 = 2\tilde{\tau}_1^2 \omega^1 \wedge \omega^3, \quad (90)$$

$$\Omega^3 = 2\tilde{\tau}_1^2 \omega^3 \wedge \omega^4, \quad (91)$$

$$\Omega^4 = 2\tilde{\tau}_1^2 \omega^1 \wedge \omega^2, \quad (92)$$

$$\Omega^5 = (5\tilde{\mu}_2 - 8)(\omega^1 + \omega^2) \wedge \omega^3 + \tilde{\tau}_1^2 (\omega^1 + \omega^2) \wedge \omega^4, \quad (93)$$

$$\Omega^6 = ((5/2)\tilde{\mu}_2 - 4)(\omega^1 - \omega^2) \wedge \omega^3 - \tilde{\tau}_1^2 (\omega^1 - \omega^2) \wedge \omega^4. \quad (94)$$

The EDS for this configuration corresponds to the following structure equations:

$$d\omega^1 = -\omega^1 \wedge \Gamma^4 + \omega^3 \wedge (\Gamma^5 - \Gamma^6) + \omega^4 \wedge \Gamma^2, \quad (95)$$

$$d\omega^2 = \omega^2 \wedge \Gamma^4 + \omega^3 \wedge (\Gamma^5 + \Gamma^6) + \omega^4 \wedge \Gamma^1, \quad (96)$$

$$d\omega^3 = \omega^1 \wedge \Gamma^1 + \omega^2 \wedge \Gamma^2 + \omega^3 \wedge \Gamma^3, \quad (97)$$

$$d\omega^4 = \omega^1 \wedge (\Gamma^5 + \Gamma^6) + \omega^2 \wedge (\Gamma^5 - \Gamma^6) - \omega^4 \wedge \Gamma^3, \quad (98)$$

$$d\Gamma^6 = \Gamma^3 \wedge \Gamma^4 - \Gamma^4 \wedge \Gamma^5 + \Omega^6 \quad (99)$$

Note that the structure equations

$$d\Gamma^\alpha = \dots, \quad \alpha = 1, \dots, 5$$

are satisfied identically, by construction. Substituting (84)-(94) into the above gives

$$d\omega^1 = \tilde{\tau}_1 \omega^1 \wedge \omega^2 - i(\tilde{\mu}_2 - 3) \omega^1 \wedge \omega^3 - i(\tilde{\mu}_2 - 2) \omega^2 \wedge \omega^3, \quad (100)$$

$$d\omega^2 = -\tilde{\tau}_1 \omega^1 \wedge \omega^2 + i(\tilde{\mu}_2 - 2) \omega^1 \wedge \omega^3 + i(\tilde{\mu}_2 - 3) \omega^2 \wedge \omega^3, \quad (101)$$

$$d\omega^3 = \tilde{\tau}_1 (\omega^1 + \omega^2) \wedge \omega^3, \quad (102)$$

$$d\omega^4 = -2i\tilde{\mu}_2 \omega^1 \wedge \omega^2 - i\nu_2 (\omega^1 - \omega^2) \wedge \omega^3 - 3\tilde{\tau}_1 (\omega^1 + \omega^2) \wedge \omega^4, \quad (103)$$

$$d\nu_2 \wedge \omega^3 = ((3i/2)\tilde{\mu}_2 (\omega^1 - \omega^2) - 3\tilde{\tau}_1 \nu_2 (\omega^1 + \omega^2) - 3\tilde{\tau}_1 \omega^4) \wedge \omega^3. \quad (104)$$

The scalar form of equations (100)-(103) is:

$$y^1_{[i,j]} = -\tilde{\tau}_1 y^1_{[i} y^2_{j]} + i(\tilde{\mu}_2 - 3) y^1_{[i} y^3_{j]} + i(\tilde{\mu}_2 - 2) y^1_{[i} y^3_{j]}, \quad (105)$$

$$y^2_{[i,j]} = \tilde{\tau}_1 y^1_{[i} y^2_{j]} - i(\tilde{\mu}_2 - 2) y^1_{[i} y^3_{j]} - i(\tilde{\mu}_2 - 3) y^1_{[i} y^3_{j]}, \quad (106)$$

$$y^3_{[i,j]} = -\tilde{\tau}_1 (y^1_{[i} y^3_{j]} + y^2_{[i} y^3_{j]}), \quad (107)$$

$$y^4_{[i,j]} = 2i\tilde{\mu}_2 y^1_{[i} y^2_{j]} + 3\tilde{\tau}_1 (y^1_{[i} y^4_{j]} + y^2_{[i} y^4_{j]}) + 2\nu_2 i (y^1_{[i} y^3_{j]} - y^2_{[i} y^3_{j]}). \quad (108)$$

The scalar form of the constrained NP equations (104) is

$$D\nu_2 = -3\tilde{\tau}_1, \quad (109)$$

$$\delta\nu_2 = -3\tilde{\tau}_1\nu_2 + (3i/2)\tilde{\mu}_2, \quad (110)$$

$$\delta^*\nu_2 = -3\tilde{\tau}_1\nu_2 - (3i/2)\tilde{\mu}_2, \quad (111)$$

where as per (4), the derivative of a scalar  $f$  is given by

$$f_{,i} = y^1_i \delta f + y^2_i \delta^* f + y^3_i \Delta f + y^4_i Df. \quad (112)$$

The scalar Bianchi equations are identically satisfied, by construction.

By inspection, equations (105)-(111) have no zeroth order integrability conditions; it isn't possible to eliminate all the derivatives from these equations. To prove involutivity, we rewrite (100)-(103) as commutator relations:

$$\delta^* \delta - \delta \delta^* = -2i\tilde{\mu}_2 D + \tilde{\tau}_1 (\delta - \delta^*), \quad (113)$$

$$\Delta \delta - \delta \Delta = i(3 - \tilde{\mu}_2) \delta + i(\tilde{\mu}_2 - 2) \delta^* + \tilde{\tau}_1 \Delta - i\nu_2 D, \quad (114)$$

$$D\delta - \delta D = -3\tilde{\tau}_1 D, \quad (115)$$

$$D\Delta - \Delta D = 0. \quad (116)$$

Combining these with (109)-(111) yields the following first-order relations:

$$3\tilde{\tau}_1 ((\delta - \delta^*)\nu_2 - 3i\tilde{\mu}_2) = 0, \quad (117)$$

$$3\tilde{\tau}_1 (D\nu_2 + 3\tilde{\tau}_1) = 0. \quad (118)$$

Since no independent first-order relations are implied, the above system of equations is involutive.

**3.3. The CH<sub>2</sub> exact solution.** Next, we integrate the structure equations (100)-(103) and describe the most general proper CH<sub>2</sub> spacetime as an exact solution. At this point, it is convenient to introduce the 1-forms

$$\theta^1 = \tilde{\tau}_1/2 (\omega^1 + \omega^2), \quad (119)$$

$$\theta^2 = -i\tilde{\tau}_1/2 (\omega^1 - \omega^2) + (2\tilde{\mu}_2 - 5)/4 \omega^3, \quad (120)$$

$$\theta^3 = \omega^3, \quad (121)$$

$$\theta^4 = (\tilde{\mu}_2/\tilde{\tau}_1)(-i/2 (\omega^1 - \omega^2) + (2\tilde{\mu}_2 - 5)/4 \omega^3) + \omega^4, \quad (122)$$

$$\theta^5 = ((1 - \tilde{\mu}_2)(\omega^1 + \omega^2) + i\Gamma^6)/\tilde{\tau}_1 = (\nu_2/\tilde{\tau}_1) \omega^3. \quad (123)$$

The structure equations now assume a particularly simple form:

$$d\theta^1 = -\theta^2 \wedge \theta^3, \quad (124)$$

$$d\theta^2 = -2\theta^1 \wedge \theta^2, \quad (125)$$

$$d\theta^3 = 2\theta^1 \wedge \theta^3, \quad (126)$$

$$d\theta^4 = -6\theta^1 \wedge \theta^4 + 2\theta^2 \wedge \theta^5, \quad (127)$$

$$d\theta^5 = -4\theta^1 \wedge \theta^5 + 3\theta^3 \wedge \theta^4. \quad (128)$$

The first 3 equations are the structure equations for  $SL_2$ . Hence,  $\theta^1, \theta^2, \theta^3$  can be integrated by means of local coordinates on  $SL_2$ . We choose the coordinatization

$$\begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix} \begin{pmatrix} e^{b/2} & 0 \\ 0 & e^{-b/2} \end{pmatrix} \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}.$$

This yields the following expressions for the Maurer-Cartan forms:

$$\theta^1 = db/2 - ae^b ds, \quad (129)$$

$$\theta^2 = da + adb - a^2 e^b ds, \quad (130)$$

$$\theta^3 = e^b ds \quad (131)$$

Substituting (123) into (128) we obtain

$$\theta^4 \equiv -(\nu_2 db + d\nu_2)/\tilde{\tau}_1 \pmod{ds}. \quad (132)$$

Writing

$$\theta^4 = -(\nu_2 db + d\nu_2)/\tilde{\tau}_1 + (2ae^b \nu_2 + e^{-3b} F) ds, \quad (133)$$

and substituting into (127) gives

$$e^{-3b} dF \wedge ds = 0.$$

Hence  $F = F(s)$ . Up to a choice of local coordinates and a choice of the function  $F(s)$ , the above solution is the most general possible. Indeed, we could take  $a, b, s, \nu_2$  as coordinates. However, it will be more convenient to set

$$\nu_2 = -3\tilde{\tau}_1 e^{-3b} t, \quad (134)$$

and to use  $a, b, s, t$  as coordinates. Finally, substituting (129)-(131), (133) into (119)-(122), we obtain the form of the general (up to a change of coordinates) solution of the  $CH_2$  field equations:

$$\omega^1 = (db/2 + i(a db + da) - e^b (a + i(a^2 + \tilde{\mu}_2/2 - 5/4)) ds) / \tilde{\tau}_1, \quad (135)$$

$$\omega^2 = (db/2 - i(a db + da) - e^b (a - i(a^2 + \tilde{\mu}_2/2 - 5/4)) ds) / \tilde{\tau}_1, \quad (136)$$

$$\omega^3 = e^b ds, \quad (137)$$

$$\begin{aligned} \omega^4 = & e^{-3b} dt - (\tilde{\mu}_2/\tilde{\tau}_1^2)(da + a db) \\ & + (F(s)e^{-3b} - 6ae^{-2b}t + (\tilde{\mu}_2/\tilde{\tau}_1^2)(a^2 + \tilde{\mu}_2/4 - 5/8)e^b) ds. \end{aligned} \quad (138)$$

**3.4. The  $CH_2$  equivalence problem and Killing vectors.** Next, we solve the local equivalence problem for the class of proper  $CH_2$  spacetimes, as described by (135) -(138). In our analysis, we use ideas from Chapter 8 of Olver[16], as well as Karlhede's algorithm. Again, it will be more convenient to work with the coframe  $\theta^a$  defined in (119) - (122). The latter differs from the canonical null-orthogonal tetrad  $\omega^a$  by a constant linear transformation, so both coframes will yield the same differential invariants.

The structure functions in equations (100)-(103) yield our first differential invariants, namely the constants  $\tilde{\tau}_1, \tilde{\mu}_1$  and the scalar  $\nu_2$ . The constrained NP equations (109) -(111) indicate that  $\delta\nu_2, \delta^*\nu_2, \Delta\nu_2$  are all functionally dependent on  $\nu_2$ . Thus the only candidate for an independent differential invariant is  $D\nu_2$ . The commutator relations (113)-(116) show that  $\delta D\nu_2, \delta^* D\nu_2, \Delta D\nu_2$  are all functionally dependent on  $\nu_2, D\nu_2$ . Proceeding inductively, we have that

$$\nu_2, D\nu_2, D^2\nu_2, D^3\nu_2 \quad (139)$$

is a maximal set of functionally independent differential invariants.

However, since we have the exact solution (135) - (138) the analysis of the equivalence problem can be considerably simplified. Let us express the first differential invariant as

$$I_1 := \nu_2/(3\tilde{\tau}_1) = e^{-3b}t.$$

Since

$$R^3 = \Gamma^{(3)} \cdot \tilde{R}^2,$$

this differential invariant arises as a component of  $R^3$  taken relative to the preferred tetrad. Working relative to the preferred tetrad, we have

$$dI_1 = -6I_1\theta^1 - e^{-4b}F(s)\theta^3 + \theta^4.$$

There are 3 cases to consider.

Case 5.2.1. Suppose that  $F(s) \neq 0$ . Let us also set

$$F_1(s) = F'(s)/F(s), \quad (140)$$

$$F_2(s) = (F_1'(s) - F_1^2(s)/8)/\sqrt{|F(s)|}. \quad (141)$$

We now have a second functionally independent differential invariant, namely

$$I_2 := \log |F| - 4b.$$

Since

$$R^4 = dR^3 + \Gamma^{(3)} \cdot R^3,$$

this is the only functionally independent invariant arising from  $R^4$ . We have

$$dI_2 = -8\theta^1 + I_3\theta^3,$$

where

$$I_3 := e^{-b}F_1(s) - 8a. \quad (142)$$

Since

$$R^5 = dR^4 + \Gamma^{(3)} \cdot R^4,$$

this is the only functionally independent invariant arising from  $R^5$ . Continuing,

$$dI_3 = -2I_3\theta^1 - 8\theta^2 + \left(I_3^2/8 + e^{I_2/2}I_4\right)\theta^3,$$

where

$$I_4 := F_2(s). \quad (143)$$

Continuing,

$$dI_4 = F_2'(s)ds = e^{-b}F_2'(s)\theta^3 = e^{I_2/4}I_5\theta^3,$$

where

$$I_5 := F_2'(s)|F(s)|^{-1/4} \quad (144)$$

is a differential invariant arising from  $R^7$ .

Now there are two subcases. Generically  $F_2'(s) \neq 0$ , and hence  $I_4$  is a functionally independent invariant, arising from  $R^6$ . Since both  $I_4$  and  $I_5$  are functions of  $s$ , locally

$$I_5 = \phi(I_4).$$

Therefore, the classification problem is solved by means of the essential constants  $\tilde{\tau}_1, \tilde{\mu}_2$  and an essential parameter function  $\phi(x)$ . Therefore, generically,  $q_M = 7$ ; the IC of our spacetime requires  $R^7$ .

Case 5.2.2. Suppose that  $I_4(s)$  is a constant. In this case,  $q_M = 6$ . The essential constants  $\tilde{\tau}_1, \tilde{\mu}_2, I_4$  solve the IC problem. Since there are only 3 functionally independent invariants. The Lie algebra of Killing vectors is 1-dimensional.

Case 5.2.3. If  $F(s) = 0$ , then the preferred tetrad possesses only one functionally independent invariant. Hence, the spacetime has a 3-dimensional isometry group, which is isomorphic to  $\mathrm{SL}_2\mathbb{R}$ . The orbits are given by  $\nu_2 = \text{const}$ . In this case,  $q_M = 4$ .

**3.5. The proper  $\mathrm{CH}_2$  spacetimes.** The above solution represents a spacetime where coupled gravity and electromagnetic waves propagate in a negatively curved background. The above proper  $\mathrm{CH}_2$  metric belongs to a general family of such spacetimes, first described in [23, 24]. However, up to now it was not known that these solutions contained a  $\mathrm{CH}_2$  subfamily. Let  $\tilde{\Lambda} < 0$  be a negative constant. Following [24], the general exact solution has the form

$$g_{ij}dx^i dx^j = 2p^{-2}d\zeta d\bar{\zeta} - 2q^2p^{-2}((-\tilde{\Lambda}A^2 + B\bar{B})r^2 + r q_s/q + 2Hp/q)ds + dr)ds, \quad (145)$$

where  $\zeta, \bar{\zeta}, r, s$  are coordinates, and where

$$p = 1 + \tilde{\Lambda}\zeta\bar{\zeta}, \quad (146)$$

$$q = (1 - \tilde{\Lambda}\zeta\bar{\zeta})A + \bar{B}\zeta + B\bar{\zeta}, \quad A = \bar{A}, \quad A = A(s), \quad B = B(s) \quad (147)$$

$$H_{\zeta\bar{\zeta}} + 2\tilde{\Lambda}p^{-2}H = f\bar{f}p/q, \quad f = f(\zeta, s), \quad (148)$$

and where

$$f d\zeta \wedge ds + \bar{f} d\bar{\zeta} \wedge ds \quad (149)$$

is the electromagnetic field. The proper  $\mathrm{CH}_2$  solution described above is a particular subclass of such spacetimes. In order to obtain the  $\mathrm{CH}_2$  specialization, one has to change coordinates and specialize the parameters of the general ansatz as follows:

$$\tilde{\Lambda} = -\tilde{\tau}_1^2, \quad A = 1, \quad B = -e^{3is}\tilde{\tau}_1, \quad (150)$$

$$H = [36 - 72/p + (27 + 16\tilde{\tau}_1^2 F(s))q/p + (10\tilde{\mu}_2 - 16)p^3/q^3]/(32\tilde{\tau}_1^2) \quad (151)$$

$$a = (\tilde{\tau}_1/p) \Im(e^{-3is}\zeta), \quad (152)$$

$$b = \log(p) - \log(q) \quad (153)$$

$$t = r + ae^b(3/2 + e^{2b}(1 + 4a^2/3))/\tilde{\tau}_1^2. \quad (154)$$

As was mentioned above, if  $F(s) = 0$ , then the metric possesses an  $\mathrm{SL}_2\mathbb{R}$  isometry group and  $q_M = 4$ . However, for generic  $F(s)$ , the specialized metric admits no Killing vectors and has IC order  $q_M = 7$ .

#### 4. CONCLUSION

We have analyzed 4-dimensional Lorentzian curvature homogeneous manifolds in terms of an exterior differential system and have proved that  $k_{1,3} = 3$ . Therefore for any 4-dimensional, Lorentzian  $M$  the  $\mathrm{CH}_3$  conditions imply that  $M$  is locally homogeneous. In addition, the class of proper  $\mathrm{CH}_2$  geometries has been explicitly determined in equations (135)–(138), these provide a counterexample to a conjecture of Gilkey stating that  $k_{1,3} = 2$ .

In regards to the invariant classification problem, it has been shown that, generically, (135)–(138) have IC order  $q_M = 7$ , thereby settling a long-standing question about the Karlhede bound for four-dimensional, Lorentz-signature spacetimes. The curvature tensor along with its first and second covariant derivatives completely fix the frame and provide two essential constants:  $\tilde{\tau}_1, \tilde{\mu}_2$ . Generically, the higher order covariant derivatives of the curvature tensor give rise to five differential invariants  $I_1, \dots, I_4, I_5 = \phi(I_4)$ . The first 4 are functionally independent. The essential

parameters  $\tilde{\tau}_1, \tilde{\mu}_2, \phi(x)$  invariantly classify the spacetime. There are 2 singular subfamilies. The parameter function  $F(s)$  is not, by itself, an invariant. However, the condition  $F(s) = 0$  is invariant. The corresponding spacetime has 3 Killing vectors. The subfamily characterized by the condition  $F'_2(s) = 0$  has 1 Killing vector.

The above  $\text{CH}_2$  solutions have constant zero-order curvature invariants and vanishing higher order curvature invariants<sup>2</sup>. Thus, these solutions are examples of constant scalar invariant spacetimes (CSI) [30]. However, it may be more natural to regard them as vanishing scalar invariant spacetimes (VSI) [31] [32] with a cosmological constant since only zeroth order invariants are nonzero constants. This indicates a slight modification required in the  $\text{CSI}_R$  conjecture [30] where we extend VSI to also include a cosmological constant. It is curious that the curvature invariants cannot be used to solve the invariant classification problem. This was also shown to occur in Einstein solvmanifolds [33].

The proper  $\text{CH}_2$  solutions describe gravitational waves and electromagnetic radiation propagating in an anti-de Sitter background. These are contained in the class of metrics presented in [24] (see also [23]). Generalizations and further analysis was subsequently given in [34] [35] [36] [37] and extended to higher dimensions in [38]. This class has been applicable in a number of important areas in the literature, such as in the study of Einstein-Yang-Mills solutions [39] which were more recently investigated to determine P-type III solutions [40]. In addition, they arise in Lovelock-Yang-Mills theory [41], in the theory of metric-affine gravity [42] [43] and in supergravity [44] [45]. A consideration of the proper  $\text{CH}_2$  class within this context may give solutions with interesting properties, or at the least may have practical relevance since the components of the curvature tensor up to its second covariant derivative is constant.

It is quite remarkable that the  $q_M = 7$  condition precisely picks out this one particular family of spacetimes with its particular physics. Note that if  $\tilde{\mu}_2 = 8/5$ , we obtain anti-de Sitter spacetime, and the choice of  $F(s)$  becomes irrelevant. However, generically the nature of  $F(s)$  must have some phenomenological interpretation in terms of the gravity and electromagnetic radiation, albeit one that requires seventh order information. Why does this not occur for flat  $\Lambda = 0$  space or for deSitter  $\Lambda > 0$  space?

In this paper we have also illustrated the applicability of exterior differential systems to the study of the curvature homogeneity problem. EDS has applications in the study of the Weyl-Lanczos problem [46] and in the analysis of vacuum solutions [47]. It is natural to expect that the use of EDS in the study of exact solutions could provide some further insights in relativity.

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